

ON IMPROPER MULTIPLE INTEGRALS*

BY

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§ 1. *Introduction.*

Two types of definitions of improper integrals

$$(1) \quad \int_{\mathfrak{A}} f(x_1, \dots, x_m) d\mathfrak{A}$$

have been employed up to the present. The first, going back to RIEMANN, CAUCHY, and in germ still farther, considers (1) as the limit of

$$\int_{\mathfrak{B}} f(x_1, \dots, x_m) d\mathfrak{B}, \quad \mathfrak{B} \doteq \mathfrak{A},$$

where $\mathfrak{B}$ is a limited partial field of $\mathfrak{A}$ in which f is finite. The most general form of this type of definition is perhaps found in the writings of JORDAN.†

The other type of definition is due to VALLÉE-POUSSIN‡. Here a truncated function $f_{n_1 n_2}(x_1, \dots, x_m)$ is introduced, whose value lies between two arbitrary numbers — n_1, n_2 . The integral (1) is now defined as the limit of

$$\int_{\mathfrak{A}} f_{n_1 n_2}(x_1, \dots, x_m) d\mathfrak{A},$$

when $n_1, n_2 \doteq \infty$.

In both the writings of JORDAN and VALLÉE-POUSSIN the field of integration $\mathfrak{A}$ is more or less restricted. The theory of JORDAN requires $\mathfrak{A}$ to have inner points; moreover much of it is essentially limited to measurable fields. The fields considered by VALLÉE-POUSSIN are even more restricted. However, a considerable portion of his work can be greatly extended without trouble.

In my paper *On Multiple Integrals* § and more fully in my book ||, I have developed a theory of proper integrals in which the field of integration $\mathfrak{A}$, is any

* Presented to the Society October 29, 1905. Received for publication November 20, 1905.

† *Journal de Mathématiques*, ser. 4, vol. 8 (1892), p. 69. Also in his *Cours d'Analyse*, vol. 2 (1894), pp. 46–95.

‡ *Journal de Mathématiques*, ser. 4, vol. 8 (1892), p. 401.

§ *These Transactions*, vol. 6 (1905), p. 416.

|| *Lectures on the Theory of Functions of Real Variables*, vol. 1, Ginn and Co., 1905. I shall refer to this as *Lectures*. Readers not having this work at hand, will find the definitions and theorems in the paper just mentioned, although less fully developed.

limited aggregate. In the present paper I wish to extend this theory to the case of improper integrals, the field $\mathfrak{A}$ being limited, or not. For the convenience of the reader I shall however treat first the case that $\mathfrak{A}$ is finite.

PART I.

FINITE FIELD OF INTEGRATION.

§ 2. Preliminary definitions and theorems.

In the following we shall suppose that one definite value is assigned to the integrand $f(x_1, \dots, x_m)$ at each point of the field of integration $\mathfrak{A}$; in such a manner, however, that the points of infinite discontinuity* of f form a discrete aggregate $\mathfrak{S}$, i. e., an aggregate of content zero. The points $\mathfrak{S}$ we shall call *singular points*. Without incurring a loss of generality we may suppose they lie in $\mathfrak{A}$.

In *Lectures*, p. 521, the terms *cell* and *division of norm δ of an aggregate* $\mathfrak{A}$ have been defined. In case $\mathfrak{A}$ embraces all the points of an m -way space $\mathfrak{R}_m$ it will be convenient in the following to limit slightly the nature of these cells by imposing the condition that each cell is complete. If now Δ be such a division of space of norm δ and $\mathfrak{A}$ an arbitrary limited aggregate, $\bar{\mathfrak{A}}_\Delta$ will denote as heretofore those cells containing points of $\mathfrak{A}$. The same symbol may, without ambiguity, denote the content of such cells. Obviously no point of $\mathfrak{A}$ can lie on the frontier of $\bar{\mathfrak{A}}_\Delta$, because it would then lie in a cell of Δ not in $\bar{\mathfrak{A}}_\Delta$, which is a contradiction.

It is sometimes convenient to subtract an aggregate $\mathfrak{B}$ from another aggregate $\mathfrak{A}$ containing $\mathfrak{B}$, in the following manner. If $\mathfrak{A}$ contains limiting points lying on the frontier of both $\mathfrak{B}$ and $\mathfrak{A} - \mathfrak{B}$, we add them to $\mathfrak{A} - \mathfrak{B}$ when not already present, and denote the result by $\mathfrak{A} \sim \mathfrak{B}$.

Another notion which we shall have to employ is the following. Let $\mathfrak{A}$, $\mathfrak{B}$ be any two aggregates. The points of $\mathfrak{A}$ in $\bar{\mathfrak{B}}_\Delta$ we denote by $\mathfrak{A}'_\delta$, while $\mathfrak{A} \sim \mathfrak{A}'_\delta$ we denote by $\mathfrak{A}_\delta$. The aggregate $\mathfrak{B}$ which we have thus used to separate $\mathfrak{A}$ into the two classes $\mathfrak{A}_\delta$, $\mathfrak{A}'_\delta$ is called the *modulus*. If we wish to denote which modulus is used in forming $\mathfrak{A}_\delta$, $\mathfrak{A}'_\delta$, a symbol as

$$\text{mod } \mathfrak{B}$$

will be added. In separating $\mathfrak{A}$ into $\mathfrak{A}_\delta$ and $\mathfrak{A}'_\delta$, frontier points not in the frontier of $\mathfrak{A}$ will usually be introduced. These frontier points common to $\mathfrak{A}_\delta$ and $\mathfrak{A}'_\delta$ may be called *the new frontier*.

If $\mathfrak{A}$, $\mathfrak{B}$ are two aggregates such that

$$\text{dist}(\mathfrak{A}, \mathfrak{B}) > 0$$

we shall say † that $\mathfrak{A}$ is *exterior to* $\mathfrak{B}$, or $\mathfrak{B}$ is exterior to $\mathfrak{A}$.

* *Lectures*, p. 212.

† *Lectures*, p. 514.

Let $\mathfrak{B}$ be a partial aggregate of $\mathfrak{A}$. If the common frontier of $\mathfrak{B}$ and $\mathfrak{A} - \mathfrak{B}$ is discrete we shall say * that $\mathfrak{B}$ is an *unmixed partial aggregate* of $\mathfrak{A}$.

If $\mathfrak{A}$ is such that for some division Δ , $\mathfrak{A}'_\delta$ is measurable, we shall say that $\mathfrak{A}$ is *relatively measurable*.

For brevity let us denote the frontier of an aggregate $\mathfrak{A}$ by

front $\mathfrak{A}$.

THEOREM 1. *Let $\mathfrak{A}$ be a limited complete aggregate. Let Δ be a division of space. Then $\mathfrak{A}$ is an inner aggregate † of $\bar{\mathfrak{A}}_\Delta$.*

For if not, the distance between the frontiers of $\mathfrak{A}$ and $\bar{\mathfrak{A}}_\Delta$ would be 0. But $\mathfrak{A}$ and $\bar{\mathfrak{A}}_\Delta$ being complete, a point of $\mathfrak{A}$ must lie on the frontier of $\bar{\mathfrak{A}}_\Delta$, which is impossible.‡

THEOREM 2. *Let Δ be a division of norm δ . Let $\mathfrak{A}, \mathfrak{B}$ be limited aggregates. Then $\mathfrak{A}_\delta, \mathfrak{A}'_\delta$, mod $\mathfrak{B}$, form an unmixed division § of $\mathfrak{A}$. If $\mathfrak{A}$ is complete, so are $\mathfrak{A}_\delta, \mathfrak{A}'_\delta$.*

For all the points common to $\mathfrak{A}_\delta, \mathfrak{A}'_\delta$ lie on the frontier $\mathfrak{F}$ of $\bar{\mathfrak{B}}_\Delta$. But $\mathfrak{F}$ is discrete. Hence these common frontier points are discrete. Moreover the common frontier points of $\mathfrak{A}_\delta, \mathfrak{A}'_\delta$ lie on $\mathfrak{F}$. The rest of the theorem is obvious.

THEOREM 3. *Let Δ, H be two divisions of space of norms δ, η . Let $\mathfrak{C}$ denote the common points of $\mathfrak{A}_\delta, \mathfrak{A}_\eta$, mod $\mathfrak{B}$; while $\alpha = \mathfrak{A}_\delta \sim \mathfrak{C}, \beta = \mathfrak{A}_\eta \sim \mathfrak{C}$. Then $\alpha, \mathfrak{C}$ and $\beta, \mathfrak{C}$ are unmixed divisions of $\mathfrak{A}_\delta, \mathfrak{A}_\eta$, respectively.*

For let $\mathfrak{f}, \mathfrak{g}$ denote the new frontier of $\mathfrak{A}_\delta, \mathfrak{A}_\eta$. Then the common frontier points of $\alpha, \mathfrak{C}$ must lie in $\mathfrak{f}, \mathfrak{g}$, and these last are discrete. Moreover the points common to $\alpha, \mathfrak{C}$ also lie in $\mathfrak{f}, \mathfrak{g}$. Hence $\alpha, \mathfrak{C}$ is an unmixed division of $\mathfrak{A}_\delta$.

THEOREM 4. *Let $f(x_1, \dots, x_m)$ be limited in $\mathfrak{A}$. Let $\mathfrak{B}_u$ be an unmixed partial aggregate of $\mathfrak{A}$ such that $\lim_{u=0} \overline{\text{cont}} \mathfrak{B}_u = \bar{\mathfrak{A}}$. Then*

$$(1) \quad \int_{\bar{\mathfrak{A}}} f = \lim_{u=0} \int_{\mathfrak{B}_u} f.$$

For let $|f| \leq M$ in $\mathfrak{A}$. Also let $\mathfrak{C}_u = \mathfrak{A} - \mathfrak{B}_u$. Then ||

$$(2) \quad \int_{\bar{\mathfrak{A}}} = \int_{\mathfrak{B}_u} + \int_{\mathfrak{C}_u}.$$

But ¶

$$\left| \int_{\mathfrak{C}_u} f \right| \leq M \overline{\text{cont}} \mathfrak{C}_u$$

is evanescent with u . Hence, passing to the limit $u = 0$ in (2), we get (1).

* Lectures, p. 519.

† Lectures, p. 515.

‡ Lectures, p. 514.

§ Lectures, p. 519.

|| Lectures, p. 534.

¶ Lectures, p. 535.

§ 3. Definition of an improper integral.

Let $f(x_1, \dots, x_m)$ be defined over the limited field $\mathfrak{A}$, while $\mathfrak{S}$ denotes the singular points as in § 2. Let Δ denote a division of space of norm δ . Then f is limited in $\mathfrak{A}_\delta$, mod $\mathfrak{S}$, by theorem 1. Hence f admits* an upper and a lower integral in $\mathfrak{A}_\delta$,

$$\bar{\int}_{\mathfrak{A}_\delta}, \quad \underline{\int}_{\mathfrak{A}_\delta}.$$

The limits

$$(1) \quad \lim_{\delta=0} \bar{\int}_{\mathfrak{A}_\delta}, \quad \lim_{\delta=0} \underline{\int}_{\mathfrak{A}_\delta},$$

are called *the upper and lower integrals of f in $\mathfrak{A}$* . They are denoted by

$$(2) \quad \bar{\int}_{\mathfrak{A}} f d\mathfrak{A}, \quad \underline{\int}_{\mathfrak{A}} f d\mathfrak{A};$$

or more shortly by

$$\bar{\int}_{\mathfrak{A}} f, \quad \underline{\int}_{\mathfrak{A}} f.$$

When the limits (1) are finite, the corresponding integrals (2) are *convergent*; we also say then that f *admits an upper and a lower integral in $\mathfrak{A}$* .

Suppose now that f is integrable in any $\mathfrak{A}_\delta$, the limit

$$(3) \quad \lim_{\delta=0} \int_{\mathfrak{A}_\delta}$$

is called *the integral of f in $\mathfrak{A}$* , and is denoted by

$$(4) \quad \int_{\mathfrak{A}} f d\mathfrak{A} \quad \text{or} \quad \int_{\mathfrak{A}} f.$$

When the limit (3) is finite we say (4) is *convergent*; we also say in this case that f is *integrable in $\mathfrak{A}$* .

Let us compare these definitions with those of JORDAN.† We observe that the fields $\mathfrak{B}$ of the auxiliary integrals employed in JORDAN's definitions are *inner, complete and measurable* partial aggregates of $\mathfrak{A}$; whereas in the definitions just given, none of these restrictions enter unless fulfilled of themselves, by the nature of $\mathfrak{A}$. To justify such restrictions some inner or inherent reasons should exist. However this may be in the case that the field of integration $\mathfrak{A}$ is itself measurable, such reasons certainly do not exist when $\mathfrak{A}$ is not measurable, as we shall see. But we can go even farther. Theoretically, fields of integration $\mathfrak{A}$ which admit partial fields $\mathfrak{B}$ of the type required by JORDAN

* Lectures, pp. 510, 528.

† Cours d'Analyse, vol. 2, p. 76.

must be regarded as exceptions. If, therefore, we are to develop a theory of integration applicable to any field, JORDAN's restrictions on the fields $\mathfrak{B}$ must be abandoned.

THEOREM 5.* *For f to admit an upper integral in $\mathfrak{A}$, it is necessary and sufficient that for each $\epsilon > 0$, there exists a division Δ of norm δ , such that*

$$(5) \quad \left| \int_{\mathfrak{B}} f \right| < \epsilon$$

for any unmixed partial aggregate $\mathfrak{B}$ in $\mathfrak{A}'$, exterior to $\mathfrak{B}$.

It is necessary. For, by hypothesis, for any $\epsilon' < \epsilon$,

$$(6) \quad \left| \int_{\mathfrak{A}_\eta} - \int_{\mathfrak{A}_\delta} \right| < \epsilon'$$

for any divisions Δ, H of norms $\delta, \eta < \delta_0$. On the division Δ let us superimpose a division of norm τ , formed by dividing the cells δ_i of $\mathfrak{A}'$ into smaller cells. Those subcells of δ_i containing only points of $\mathfrak{A}$ belonging to $\mathfrak{B}$, let us group together into a single cell δ'_i . The other points of δ_i form a cell δ''_i . Thus each cell δ_i falls into two cells δ'_i, δ''_i . Let E be the division which splits up the cells δ_i in this way. Let H be the division formed by superimposing E on Δ . Let $\mathfrak{B}_\tau$ denote the points of $\mathfrak{B}$ in the cells δ'_i . Then

$$\int_{\mathfrak{A}_\eta} = \int_{\mathfrak{A}_\delta} + \int_{\mathfrak{B}_\tau}.$$

This in (6) gives

$$\left| \int_{\mathfrak{B}_\tau} \right| < \epsilon'.$$

Let now $\tau \doteq 0$. Obviously

$$\overline{\text{cont } \mathfrak{B}_\tau} \doteq \overline{\mathfrak{B}}.$$

Hence by theorem 4

$$\left| \int_{\mathfrak{B}} \right| \leq \epsilon' < \epsilon$$

which is (5).

It is sufficient. For, suppose the relation (5) is satisfied for the division Δ . Let H, H' be two other divisions of norms η, η' . Let $\mathfrak{C}$ denote the common points of $\mathfrak{A}_\eta, \mathfrak{A}_{\eta'}$, while $\alpha = \mathfrak{A}_\eta \sim \mathfrak{C}, \alpha' = \mathfrak{A}_{\eta'} \sim \mathfrak{C}$. Then if η, η' are taken sufficiently small, say $< \eta_0$, α, α' will lie in $\mathfrak{A}'_\delta$. We have now

$$\int_{\mathfrak{A}_\eta} = \int_{\mathfrak{C}} + \int_{\alpha}, \quad \int_{\mathfrak{A}_{\eta'}} = \int_{\mathfrak{C}} + \int_{\alpha'}.$$

Hence

$$\left| \int_{\mathfrak{A}_\eta} - \int_{\mathfrak{A}_{\eta'}} \right| = \left| \int_{\alpha} - \int_{\alpha'} \right| \leq \left| \int_{\alpha} \right| + \left| \int_{\alpha'} \right| < 2\epsilon.$$

* A similar theorem holds for the lower integral.

The integrals

$$\bar{\int}_{\mathfrak{A}'_{\delta}} f \quad \underline{\int}_{\mathfrak{A}'_{\delta}} f$$

are respectively the *upper and lower singular integrals*, of norm δ .

Let H be a division of norm η . Let $\mathfrak{B} = \mathfrak{A}'_{\delta}$. The points of $\mathfrak{B}_{\eta}$ we shall also denote by $\mathfrak{A}'_{\delta, \eta}$.

The integrals

$$\bar{\int}_{\mathfrak{A}'_{\delta, \eta}} f \quad \underline{\int}_{\mathfrak{A}'_{\delta, \eta}} f$$

will be called the *incomplete singular integrals*. In contradistinction the former may be called *complete*. From theorem 5 we have at once:

THEOREM 6. *For f to admit an upper or lower integral in $\mathfrak{A}$, it is necessary and sufficient that the corresponding singular integral, complete or incomplete, is evanescent with its norm.*

§ 4. Elementary properties of improper integrals.

THEOREM 7.* *If f admits an upper integral in $\mathfrak{A}$, it admits an upper integral in any unmixed partial aggregate $\mathfrak{B}$ of $\mathfrak{A}$. Moreover, for each $\sigma > 0$, there exists a δ_0 , such that*

$$(1) \quad \left| \bar{\int}_{\mathfrak{B}} f \right| < \sigma,$$

if $\mathfrak{B}$ lies in $\mathfrak{A}'_{\delta}$, $\delta \leq \delta_0$.

To show the convergence of the upper integral of f in $\mathfrak{B}$, we need only show that for any division H of norm $\eta \leq \eta_0$,

$$\left| \bar{\int}_{\mathfrak{B}'_{\eta}} f \right| < \epsilon.$$

But $\mathfrak{B}'_{\eta}$ is an unmixed partial aggregate of $\mathfrak{A}'_{\delta}$. We can now apply theorem 5. The second half of theorem also follows from theorem 5. For there exists a division Δ of norm δ , such that for any $0 < \sigma' < \sigma$

$$\left| \bar{\int}_{\mathfrak{A}'_{\delta}} f \right| < \sigma'.$$

Passing to the limit $\delta = 0$ we get (1).

THEOREM 8.* *If f admits an upper integral in $\mathfrak{A}$, there exists for each $\epsilon > 0$, a $\sigma > 0$, such that*

$$\left| \bar{\int}_{\mathfrak{B}} f \right| < \epsilon$$

* A similar theorem holds for the lower integral.

for any partial unmixed aggregate $\mathfrak{B}$ of $\mathfrak{A}$, whose upper content does not exceed σ .

For, by theorem 7, there exists a division Δ such that

$$\left| \bar{\int}_{\mathfrak{B}_\delta'} \right| < \frac{\epsilon}{2}.$$

Now f being limited in $\mathfrak{A}_\delta$, let $|f| \leq M$ in $\mathfrak{A}_\delta$. Then

$$\left| \bar{\int}_{\mathfrak{B}_\delta} \right| \leq M \overline{\text{cont}} \mathfrak{B}_\delta \leq M \bar{\mathfrak{B}}.$$

Let us take

$$\sigma < \frac{\epsilon}{2M}, \quad \bar{\mathfrak{B}} \leq \sigma.$$

Then

$$\left| \bar{\int}_{\mathfrak{B}} \right| \leq \left| \bar{\int}_{\mathfrak{B}_\delta} \right| + \left| \bar{\int}_{\mathfrak{B}_\delta'} \right| < \frac{\epsilon}{2} + \frac{\epsilon}{2}.$$

THEOREM 9.* Let $\mathfrak{A}_1, \mathfrak{A}_2, \dots, \mathfrak{A}_n$ be an unmixed division of $\mathfrak{A}$. If f admits an upper integral in $\mathfrak{A}$, or in the fields $\mathfrak{A}_1, \dots, \mathfrak{A}_n$,

$$(2) \quad \bar{\int}_{\mathfrak{A}} f = \bar{\int}_{\mathfrak{A}_1} f + \dots + \bar{\int}_{\mathfrak{A}_n} f.$$

For, suppose f admits an upper integral in the partial fields $\mathfrak{A}_1, \dots, \mathfrak{A}_n$. Let Δ be a division of norm δ . Then $\mathfrak{A}_{1,\delta}, \dots, \mathfrak{A}_{n,\delta}$, are unmixed divisions of $\mathfrak{A}_\delta$. Hence †

$$(3) \quad \bar{\int}_{\mathfrak{A}_\delta} = \bar{\int}_{\mathfrak{A}_{1,\delta}} + \dots + \bar{\int}_{\mathfrak{A}_{n,\delta}}.$$

Since by hypothesis the limits on the right for $\delta = 0$ exist we have (2) on passing to the limit in (3). If we suppose, secondly, that f admits an upper integral in $\mathfrak{A}$, it admits an upper integral in each of the fields $\mathfrak{A}_\kappa$ by theorem 7, and we are brought back to the case just treated.

By a similar passage to the limit, a number of theorems analogous to theorems in proper integrals may be at once established. For example the following:

THEOREM 10.* Let $f_1, f_2, \dots, f_n$ admit upper integrals in $\mathfrak{A}$. Let $c_1, c_2, \dots, c_n \geq 0$. Then

$$\bar{\int}_{\mathfrak{A}} \sum c_\kappa f_\kappa \leq \sum c_\kappa \bar{\int}_{\mathfrak{A}} f_\kappa.$$

* A similar theorem holds for the lower integrals.

† Lectures, p. 534.

THEOREM 11.* If $c < 0$ and f admits a lower or an upper integral in $\mathfrak{A}$

$$\int_{\mathfrak{A}} cf = c \int_{\mathfrak{A}} f.$$

THEOREM 12.* Let f, g admit upper integrals in $\mathfrak{A}$. If $f = g$ except at points of a discrete aggregate,

$$\bar{\int}_{\mathfrak{A}} f = \bar{\int}_{\mathfrak{A}} g.$$

THEOREM 13.* Let f, g admit upper integrals in $\mathfrak{A}$ and let $f \geq g$ except at points of a discrete aggregate. Then

$$\bar{\int}_{\mathfrak{A}} f \geq \bar{\int}_{\mathfrak{A}} g.$$

THEOREM 14.* Let f admit an upper integral in $\mathfrak{A}$. Let $\mathfrak{B}_u$ be an unmixed partial aggregate of $\mathfrak{A}$ such that $\overline{\text{cont}} \mathfrak{B}_u \doteq \bar{\mathfrak{A}}$, as $u \doteq 0$. Then

$$\bar{\int}_{\mathfrak{A}} f = \lim_{u=0} \bar{\int}_{\mathfrak{B}_u} f.$$

For, setting $\mathfrak{C}_u = \mathfrak{A} - \mathfrak{B}_u$,

$$\bar{\int}_{\mathfrak{A}} = \bar{\int}_{\mathfrak{B}_u} + \bar{\int}_{\mathfrak{C}_u}.$$

We pass now to the limit $u = 0$, using theorem 8.

THEOREM 15.* Let $f \geq 0$ in $\mathfrak{A}$. Let $\mathfrak{B}_1, \mathfrak{B}_2, \dots$, be a sequence of unmixed partial aggregates of $\mathfrak{A}$, exterior to $\mathfrak{J}$, such 1° that each $\mathfrak{B}_n$ contains the preceding $\mathfrak{B}_{n-1}$ and 2° that $\lim \bar{\mathfrak{B}}_n = \bar{\mathfrak{A}}$. Then f admits an upper integral in $\mathfrak{A}$; moreover

$$(3) \quad \bar{\int}_{\mathfrak{A}} f = \lim_{n=\infty} \bar{\int}_{\mathfrak{B}_n} f$$

provided the integrals on the right remain less than some fixed M .

Without loss of generality, we may suppose that any given $\mathfrak{A}_s$ lies in some $\mathfrak{B}_n$, n being taken large enough. In fact the points in $\mathfrak{A}_s$ not in some $\mathfrak{B}_n$ must be discrete. If we add them to $\mathfrak{B}_n$, the integral on the right of (3) remains unaltered. Next we observe that the limit on the right of (3) must exist, since

$$\bar{\int}_{\mathfrak{B}_1}, \bar{\int}_{\mathfrak{B}_2}, \dots$$

is a limited monotone increasing sequence. Hence

* A similar theorem holds for the lower integral.

$$(4) \quad \int_{\mathfrak{B}_\nu} - \int_{\mathfrak{B}_n} < \epsilon \quad (\nu > n)$$

if n is taken sufficiently large.

Now to show that f admits an upper integral in $\mathfrak{A}$, we have only to show that for some η_0 ,

$$(5) \quad \left| \bar{\int}_{\mathfrak{A}_\eta} - \bar{\int}_{\mathfrak{A}_{\eta'}} \right| < \epsilon \quad (\eta, \eta' < \eta_0).$$

But let us take η_0 so small that $\mathfrak{A}_\eta, \mathfrak{A}_{\eta'}$, both contain $\mathfrak{B}_n$, while ν is taken so large that $\mathfrak{B}_\nu$ contain $\mathfrak{A}_\eta, \mathfrak{A}_{\eta'}$. Then (5) follows at once from (4). Thus f admits an upper integral in $\mathfrak{A}$. The rest of the theorem follows from theorem 14.

§ 5. Absolutely convergent improper integrals.

THEOREM 16. *Let the upper integral of $|f|$ in $\mathfrak{A}$ be convergent. Then the lower integral of $|f|$ and the lower and upper integrals of f are convergent in $\mathfrak{A}$.*

For, in any $\mathfrak{A}_\delta$,

$$0 \leq \int_{\mathfrak{A}_\delta} |f| \leq \bar{\int}_{\mathfrak{A}_\delta} |f|.$$

Thus by theorem 15, the lower integral of $|f|$ in $\mathfrak{A}$ is convergent.

For brevity let us set $\mathfrak{A}_\delta = \mathfrak{B}$. Then for any division H , of norm η ,

$$\left| \bar{\int}_{\mathfrak{B}_\eta} f \right| \leq \bar{\int}_{\mathfrak{B}_\eta} |f| \leq \bar{\int}_{\mathfrak{B}} |f|.$$

But we may take δ so small that the integral on the right is $< \epsilon$. Hence by theorem 6, the upper integral of f is convergent in $\mathfrak{A}$. Similarly, the lower integral of f is convergent.

When the integrals

$$\bar{\int}_{\mathfrak{A}} |f|, \quad \int_{\mathfrak{A}} |f|, \quad \int_{\mathfrak{A}} |f|,$$

are convergent, the corresponding integrals

$$\bar{\int}_{\mathfrak{A}} f, \quad \int_{\mathfrak{A}} f, \quad \int_{\mathfrak{A}} f,$$

are said to be *absolutely convergent*.

THEOREM 17. *Let the upper integral of f in $\mathfrak{A}$ be absolutely convergent. Then the upper integral of f in any* partial field $\mathfrak{B}$ of $\mathfrak{A}$ is absolutely convergent.*

* The reader should note that $\mathfrak{B}$ is not necessarily an unmixed partial aggregate of $\mathfrak{A}$.

For, by hypothesis,

$$\int_{\mathfrak{A}_\delta'} |f| < \epsilon \quad (\delta \leq \delta_0).$$

But, by *Lectures*, p. 535,

$$\int_{\mathfrak{A}'_{\delta, \eta}} |f| \leq \int_{\mathfrak{A}'_\delta} |f| \leq \int_{\mathfrak{A}'_\delta} |f| < \epsilon.$$

Hence by theorem 6, $|f|$ admits an upper integral in $\mathfrak{B}$.

A number of theorems follow directly now, on passing to the limit, from the corresponding theorems on proper integrals. We note the following:

THEOREM 18. *Let the upper integral of f in $\mathfrak{A}$ be absolutely convergent. Then*

$$\int_{\mathfrak{A}} f \leq \left| \int_{\mathfrak{A}} f \right| \leq \int_{\mathfrak{A}} |f|.$$

THEOREM 19. *Let the upper integral of f in $\mathfrak{A}$ be absolutely convergent. Then*

$$-\int_{\mathfrak{A}} |f| \leq \int_{\mathfrak{A}} f \leq \int_{\mathfrak{A}} |f|.$$

For, from

$$-|f| \leq f \leq |f|$$

we have

$$\int_{\mathfrak{A}} -|f| \leq \int_{\mathfrak{A}} f \leq \int_{\mathfrak{A}} |f|.$$

THEOREM 20. *Let the upper integral of f in $\mathfrak{A}$ be absolutely convergent. Then*

$$-\int_{\mathfrak{A}} |f| \leq \int_{\mathfrak{A}} f \leq \int_{\mathfrak{A}} |f|.$$

THEOREM 21. *Let $\mathfrak{A}$ be relatively measurable. If*

$$\int_{\mathfrak{A}} f, \quad \int_{\mathfrak{A}} f$$

are both convergent, they are absolutely convergent.

For,* let $g = f$ when $f > 0$, and 0 when $f \leq 0$. Let $h = -f$ when $f < 0$, and 0 when $f \geq 0$. Then

$$|f| = g + h.$$

Thus the upper integral of $|f|$ will be convergent, if the upper integrals of g and h are. Let us show the upper integral of g is convergent in $\mathfrak{A}$. Since $\mathfrak{A}$

* Cf. JORDAN, *Cours d'Analyse*, vol. 2, p. 78, § 76.

is relatively measurable, there exists a division Δ such that $\mathfrak{B} = \mathfrak{U}'_\delta$ is measurable. Let H be a division of $\mathfrak{B}$ of norm η . Then employing the customary notation, we can choose η_0 so small that

$$(1) \quad 0 \leq \sum_{\kappa} M_{\kappa} \delta_{\kappa} - \int_{\mathfrak{B}_\eta} f < \frac{\epsilon}{2} \quad (\eta \leq \eta_0),$$

the summation extending over those cells δ_{κ} of H containing points of $\mathfrak{B}_\eta$.

But

$$\int_{\mathfrak{B}_\eta} = \sum_{\kappa} \int_{\delta_{\kappa}},$$

Hence (1) gives

$$(2) \quad \sum_{\kappa} \left\{ M_{\kappa} \delta_{\kappa} - \int_{\delta_{\kappa}} f \right\} < \frac{\epsilon}{2}.$$

Since each term in the bracket is ≥ 0 , this relation is still true when the summation is extended over only a part of the cells δ_{κ} . Let now δ_i denote those cells in which $f > 0$ at some point in each of them. Then (2) gives

$$\sum_i M_i \delta_i < \sum_i \int_{\delta_i} f + \frac{\epsilon}{2}.$$

Let N_i denote the maximum of g in δ_i . Then

$$\int_{\mathfrak{B}_\eta} g \leq \sum_i N_i \delta_i,$$

by *Lectures*, p. 530. But in δ_i , $N_i = M_i$. Hence

$$\int_{\mathfrak{B}_\eta} g \leq \sum_i M_i \delta_i < \sum_i \int_{\delta_i} f + \frac{\epsilon}{2} = \int_{\mathfrak{C}} f + \frac{\epsilon}{2},$$

where $\mathfrak{C}$ is the sum of the cells δ_i .

By theorem 7 we can take δ so small that the integral on the right is numerically $< \epsilon/2$. Then

$$\left| \int_{\mathfrak{B}_\eta} g \right| < \epsilon.$$

Hence the upper integral of g in $\mathfrak{A}$ is convergent by theorem 6.

To show that the upper integral of h is convergent in $\mathfrak{A}$ we need only observe that

$$\int_{\mathfrak{A}} f = - \int_{\mathfrak{A}} (-f);$$

the integral on the right being convergent by hypothesis. But the foregoing reasoning on f and g is obviously applicable to $-f$ and h .

To show now that the upper integral of $|f|$ in $\mathfrak{A}$ is convergent, we observe that

$$\int_{\mathfrak{x}_\eta} \bar{|f|} = \int_{\mathfrak{x}_\eta} (g + h) \leq \int_{\mathfrak{x}_\eta} g + \int_{\mathfrak{x}_\eta} h < \epsilon \quad (\delta \leq \delta_0),$$

if δ_0 is taken small enough.

From theorems 16 and 21 we have:

THEOREM 22. *For the upper and lower integrals over a measurable field $\mathfrak{A}$ to be simultaneously convergent, it is necessary and sufficient that the upper integral is absolutely convergent in $\mathfrak{A}$.*

THEOREM 23. *For f to be integrable in the relatively measurable field $\mathfrak{A}$, it is necessary and sufficient that f is absolutely integrable in $\mathfrak{A}$.*

§ 5. Reduction of multiple integrals to multiply iterated integrals.

Let us adopt the notation of *Lectures*, pp. 524, 525, dropping for brevity the index i from $x_i, \mathfrak{x}_i, \mathfrak{X}_i, \mathfrak{P}_i, \dots$. With respect to $\mathfrak{S}$ as modulus, any division of space of norm δ divides each α and $\mathfrak{P}$ into two sets $\alpha_\delta, \alpha'_\delta, \mathfrak{P}_\delta, \mathfrak{P}'_\delta$, just as it divides the given aggregate $\mathfrak{A}$ into $\mathfrak{A}_\delta, \mathfrak{A}'_\delta$.

We wish now to introduce a division of space which we shall call *cylindrical*. Let D be any division of norm d , of the plane Π . Through the frontier of the cells of D we pass lines parallel to the x axis, generating a system of cylinders. These cylinders may now be divided up in any way, so as to effect a division Δ of space of norm $\delta \leq d$. In particular, whenever desirable, we may suppose these cylinders divided by passing planes perpendicular to the x axis, effecting thereby any prescribed division of the x axis. Instead of starting with a division of the plane Π , we may start with a division of the axis x .

We have now to consider more carefully how the integrals

$$(1) \quad \int_{\mathfrak{P}} \bar{f}, \quad \int_{\mathfrak{P}} f, \quad \int_{\alpha} \bar{f}, \quad \int_{\alpha} f$$

converge.* Let

$$(2) \quad \Delta_1, \Delta_2, \dots$$

be a sequence of cylindrical divisions of norms $\delta_1, \delta_2, \dots \doteq 0$. If for each $0 < \epsilon < 1$, there exists an m , such that, for each x in $\mathfrak{x}$

$$(4) \quad \left| \int_{\mathfrak{P}'_{\delta_n}} \bar{f} \right| < \epsilon,$$

for any $n \geq m$, we shall say that

$$(5) \quad \int_{\mathfrak{P}} \bar{f}$$

is uniform in $\mathfrak{x}$. Similar definitions apply to the other integrals (1).

* Cf. VALLÉE-POUSSIN, loc. cit., p. 435 seq.

Let b_ϵ be a discrete partial aggregate of $\mathfrak{x}$ depending on ϵ . Let D be a division of the x axis of norm d , into intervals. Let $\mathfrak{x}_\epsilon$ denote those points of $\mathfrak{x}$ which lie in intervals containing no point of b_ϵ , while $\mathfrak{x}'_\epsilon$ may denote the other points of $\mathfrak{x}$. If for each ϵ , the inequalities (4) hold while x ranges over $\mathfrak{x}_\epsilon$, D being taken at pleasure but fixed, we say that (5) is *regular in $\mathfrak{x}$* . Similar definitions apply to the other integrals in (1).

For brevity we shall replace the symbols $\mathfrak{A}_{\delta_n}, \mathfrak{P}_{\delta_n}, \dots$ by $\mathfrak{A}_n, \mathfrak{P}_n, \dots$. The points of $\mathfrak{A}, \mathfrak{A}_n$, whose projections fall in $\mathfrak{x}_\epsilon$, we denote by $\mathfrak{A}_\epsilon, \mathfrak{A}_{n,\epsilon}$. In general, the projection $\mathfrak{x}_n$ of $\mathfrak{P}_n$ on the x axis will not embrace all the points of $\mathfrak{x}$. When this is the case we may adjoin a discrete set of points to our field of integration $\mathfrak{A}$, whose projections are the missing points of $\mathfrak{x}$. At these new points we may give f the value of 0. In doing this we have changed neither the value nor the character of the integrals involved. We may therefore suppose $\mathfrak{x}_n = \mathfrak{x}$ without loss of generality. Obviously similar remarks apply to the other integrals of (1).

THEOREM 24. *If one of the integrals (1) is uniform, it is limited.*

For example, suppose (5) is uniform in $\mathfrak{x}$. Then there exists an m , such that

$$\int_{\mathfrak{P}} f = \int_{\mathfrak{P}_m} f + \int_{\mathfrak{P}'_m} f = \int_{\mathfrak{P}_m} f + \epsilon' \quad (|\epsilon'| < 1).$$

But f being limited in $\mathfrak{A}_m$, the integral on the right is numerically less than some M . Hence

$$(6) \quad \left| \int_{\mathfrak{P}} f \right| < M + 1, \text{ in } \mathfrak{x}.$$

THEOREM 25. *Let one of the integrals (1) be regular. Then its points of infinite discontinuity form a discrete aggregate.*

For example, let (5) be regular in $\mathfrak{x}$. Then (6) holds in $\mathfrak{x}_\epsilon$, however D be chosen. Suppose now i were not discrete. Then

$$\bar{i}_D > p,$$

where p is some positive number. We can choose D such that

$$\bar{\mathfrak{x}}_{\epsilon, D} < p,$$

which is a contradiction, since

$$\bar{\mathfrak{x}}_{\epsilon, D} \geq \bar{i}_D.$$

THEOREM 26. *Let f admit an upper and lower integral in the measurable field $\mathfrak{A}$. Let the integrals **

$$(7) \quad \int_{\mathfrak{P}} f, \quad \int_{\mathfrak{P}} f$$

* If only one of these integrals is uniform the term involving the other will drop out of (8).

be uniform in $\mathfrak{x}$. Then

$$(8) \quad \int_{\mathfrak{x}} f \leq \int_{\mathfrak{x}} \int_{\mathfrak{y}} f \leq \int_{\mathfrak{x}} \bar{\int}_{\mathfrak{y}} f \leq \bar{\int}_{\mathfrak{x}} f.$$

For, since (7) are uniform, for each $\epsilon > 0$ there exists an m such that for any $n \geq m$

$$\bar{\int}_{\mathfrak{y}} f = \bar{\int}_{\mathfrak{y}_m} f + \epsilon' \quad (|\epsilon'| < \epsilon).$$

Hence, by *Lectures*, p. 535,

$$\int_{\mathfrak{x}} \int_{\mathfrak{y}_n} f + \int_{\mathfrak{x}} \epsilon' \leq \int_{\mathfrak{x}} \int_{\mathfrak{y}} f \leq \int_{\mathfrak{x}} \bar{\int}_{\mathfrak{y}} f \leq \int_{\mathfrak{x}} \bar{\int}_{\mathfrak{y}_n} f + \int_{\mathfrak{x}} \epsilon'.$$

But by *Lectures*, p. 538,

$$\int_{\mathfrak{x}_n} f \leq \int_{\mathfrak{x}} \int_{\mathfrak{y}_n} f \leq \int_{\mathfrak{x}} \bar{\int}_{\mathfrak{y}_n} f \leq \bar{\int}_{\mathfrak{x}_n} f.$$

Hence, A denoting a positive constant,

$$-\epsilon A + \int_{\mathfrak{x}_n} f \leq \int_{\mathfrak{x}} \int_{\mathfrak{y}} f \leq \int_{\mathfrak{x}} \bar{\int}_{\mathfrak{y}} f \leq \bar{\int}_{\mathfrak{x}_n} f + \epsilon A.$$

As $\epsilon \doteq 0$, $n \doteq \infty$. The relation (8) follows now from theorem 14, on passing to the limit.

THEOREM 27. *Let f admit a lower and upper integral in the measurable field $\mathfrak{A}$. Let the integrals**

$$\bar{\int}_{\mathfrak{y}} f$$

be regular and admit lower and upper integrals in $\mathfrak{x}$. Then

$$(9) \quad \int_{\mathfrak{x}} f \leq \int_{\mathfrak{x}} \int_{\mathfrak{y}} f \leq \int_{\mathfrak{x}} \bar{\int}_{\mathfrak{y}} f \leq \bar{\int}_{\mathfrak{x}} f.$$

For the reasoning of theorem 26 shows that for each ϵ ,

$$-\epsilon A + \int_{\mathfrak{x}_{n,\epsilon}} f \leq \int_{\mathfrak{x}_\epsilon} \int_{\mathfrak{y}} f \leq \int_{\mathfrak{x}_\epsilon} \bar{\int}_{\mathfrak{y}} f \leq \bar{\int}_{\mathfrak{x}_{n,\epsilon}} f + \epsilon A$$

for any $n \geq m$. Let $n \doteq \infty$. Then by theorem 14

$$-\epsilon A + \int_{\mathfrak{x}_\epsilon} f \leq \int_{\mathfrak{x}_\epsilon} \int_{\mathfrak{y}} f \leq \int_{\mathfrak{x}_\epsilon} \bar{\int}_{\mathfrak{y}} f \leq \bar{\int}_{\mathfrak{x}_\epsilon} f + \epsilon A.$$

Letting now $\epsilon \doteq 0$, we have (9), since ϵ is small at pleasure.

* Cf. footnote to theorem 26.

The theorems 26, 27 may be modified in a variety of ways which the attentive reader will readily perceive. For lack of space, we must leave them unmentioned, excepting one important case, which results in replacing the integrals

$$\int_{\mathfrak{B}} f \quad \text{by} \quad \int_a f.$$

The whole theory of inversion in multiply iterated integrals is a consequence of these relations, too obvious to need farther development.

PART II.

INFINITE FIELD OF INTEGRATION.

§ 6. *Definitions. Elementary properties.*

Let $\mathfrak{A}$ be an unlimited point aggregate over which the one valued function f is defined.

Let $\mathfrak{A}_r$ denote the points of $\mathfrak{A}$ whose distance from the origin is $\leq r$ while $\mathfrak{A}'$ may denote the other points of $\mathfrak{A}$. Let the singular points Δ of f be discrete in any $\mathfrak{A}_r$. Let

$$\int_a f$$

be convergent in any limited unmixed partial aggregate α in $\mathfrak{A}$.

Let $\mathfrak{A}_r^*$ denote a limited unmixed partial aggregate containing at least all the points of $\mathfrak{A}_r$, and in general other points of $\mathfrak{A}$. If

$$(1) \quad \lim_{r=\infty} \int_{\mathfrak{A}_r^*} f$$

exists, finite or infinite, we denote it by

$$(2) \quad \int_{\mathfrak{A}} f$$

and call it the *upper integral of f in $\mathfrak{A}$* . When (1) is finite we say (2) is *convergent in $\mathfrak{A}$* , and f admits an upper integral in $\mathfrak{A}$. In a similar manner we define the symbols

$$\int_{\mathfrak{A}} f, \quad \int_{\mathfrak{A}} f \quad (\text{Cf. § 3}).$$

THEOREM 28.* *For f to admit an upper integral in $\mathfrak{A}$, it is necessary and*

* A similar theorem holds for the lower integral.

sufficient that for each $\epsilon > 0$, there exists an r , such that

$$\left| \int_{\mathfrak{B}} \bar{f} \right| < \epsilon$$

for any unmixed partial aggregate $\mathfrak{B}$ of $\mathfrak{A}'_r$.

The demonstration is obvious. A large number of theorems for improper integrals, whose field of integration $\mathfrak{A}$ is limited, hold also when $\mathfrak{A}$ is unlimited. So, for example, the theorems 9 to 13 and 16 to 23 inclusive. It is, of course, necessary to define the terms measurable, unmixed and discrete, for unlimited aggregates. This we do by simply requiring that these properties hold in any $\mathfrak{A}_r$. The theorems 14, 15, also hold, if their enunciation is slightly modified.

§ 7. Iterated integrals.

Either or both the projections $\mathfrak{r}$, $\mathfrak{X}$, may be unlimited. To fix the ideas, let $\mathfrak{r}$ be limited. We shall say that

$$(1) \quad \int_{\mathfrak{B}} \bar{f}$$

is *uniform in $\mathfrak{r}$* , provided:

1°.

$$(2) \quad \int_{\mathfrak{B}_r} \bar{f}$$

is uniform in $\mathfrak{r}$ for any r .

2°. For each $\epsilon > 0$, there exists an r_0 , such that

$$(3) \quad \left| \int_{\mathfrak{B}'_r} \bar{f} \right| < \epsilon \quad (r \geq r_0)$$

for any point of $\mathfrak{r}$.

We shall say that (1) is *regular in $\mathfrak{r}$* , provided:

1°. The integral (2) is regular in $\mathfrak{r}$ for any r .

2°. The relation (3) holds in $\mathfrak{r}_\epsilon$, the division D being chosen at pleasure.

Cf. § 5.

Similar definitions hold for the other integrals in

$$\int_{\mathfrak{B}} \bar{f}, \quad \int_{\mathfrak{B}} f.$$

When $\mathfrak{X}$ is limited, $\mathfrak{r}$ being unlimited, we have similar definitions for the integrals

$$\int_{\mathfrak{a}} \bar{f} \quad \int_{\mathfrak{a}} f$$

with respect to $\mathfrak{X}$.

Obviously the theorems 24, 25 hold in the present case, when $\mathfrak{r}$ or $\mathfrak{X}$ is limited. The extension of theorem 26, 27 to the case that either $\mathfrak{r}$ or $\mathfrak{X}$ or both are unlimited presents no difficulty. To indicate how this may be done, we enunciate and prove the following two theorems.

THEOREM 29. *Let f admit lower and upper integrals in the infinite measurable field $\mathfrak{A}$, whose projection $\mathfrak{r}$ is limited. Let*

$$\int_{\mathfrak{B}}^{\bar{}} f$$

be uniform in $\mathfrak{r}$. Then

$$(5) \quad \int_{\mathfrak{A}} f \equiv \int_{\mathfrak{r}} \int_{\mathfrak{B}}^{\bar{}} f \equiv \int_{\mathfrak{r}} \int_{\mathfrak{B}}^{\bar{}} f \equiv \int_{\mathfrak{A}}^{\bar{}} f.$$

If $\mathfrak{r}$ is also infinite the relation (5) still holds if the integrals (4) are uniform in any $\mathfrak{r}$, and admit lower and upper integrals in $\mathfrak{r}$.

Let us suppose first that $\mathfrak{r}$ is limited. Since (4) are uniform,

$$\int_{\mathfrak{B}}^{\bar{}} = \int_{\mathfrak{B}_r}^{\bar{}} + \epsilon' \quad (|\epsilon'| < \epsilon).$$

Hence

$$\int_{\mathfrak{r}} \int_{\mathfrak{B}_r}^{\bar{}} + \int_{\mathfrak{r}} \epsilon' \equiv \int_{\mathfrak{r}} \int_{\mathfrak{B}}^{\bar{}} \equiv \int_{\mathfrak{r}} \int_{\mathfrak{B}}^{\bar{}} \equiv \int_{\mathfrak{r}} \int_{\mathfrak{B}}^{\bar{}} + \int_{\mathfrak{r}} \epsilon'.$$

Then by theorem 26, A denoting a positive constant,

$$-\epsilon A + \int_{\mathfrak{r}} \epsilon' \equiv \int_{\mathfrak{r}} \int_{\mathfrak{B}}^{\bar{}} \equiv \int_{\mathfrak{r}} \int_{\mathfrak{B}}^{\bar{}} \equiv \int_{\mathfrak{r}}^{\bar{}} + \epsilon A.$$

Letting $\epsilon \doteq 0$, and $r \doteq \infty$ we get (5).

Suppose now $\mathfrak{r}$ infinite. The relation (5) holding for any limited $\mathfrak{r}$, holds in the limit, when $\mathfrak{r}$ is infinite, since all the integrals involved exist by hypothesis.

THEOREM 30. *Let f admit lower and upper integrals in the infinite measurable field $\mathfrak{A}$, whose projection $\mathfrak{r}$ is limited. Let*

$$(6) \quad \int_{\mathfrak{B}}^{\bar{}} f$$

be regular, and admit lower and upper integrals in $\mathfrak{r}$. Then

$$(7) \quad \int_{\mathfrak{A}} f \equiv \int_{\mathfrak{r}} \int_{\mathfrak{B}}^{\bar{}} f \equiv \int_{\mathfrak{r}} \int_{\mathfrak{B}}^{\bar{}} f \equiv \int_{\mathfrak{A}}^{\bar{}} f.$$

If $\mathfrak{r}$ is also infinite, the relation (7) still holds, if the integrals (6) are regular in any $\mathfrak{r}$, and admit lower and upper integrals in $\mathfrak{r}$.

Let us suppose first that $\mathfrak{r}$ is limited. Then by hypothesis

$$\int_{\mathfrak{z}}^{\bar{\mathfrak{z}}} = \int_{\mathfrak{z}_r}^{\bar{\mathfrak{z}}} + \epsilon' \quad (|\epsilon'| < \epsilon)$$

for any point of $\mathfrak{r}_\epsilon$. Hence

$$\int_{\mathfrak{r}_\epsilon}^{\bar{\mathfrak{r}}} + \int_{\mathfrak{r}_\epsilon}^{\bar{\mathfrak{r}}} \epsilon' \leq \int_{\mathfrak{r}_\epsilon}^{\bar{\mathfrak{r}}} \int_{\mathfrak{z}}^{\bar{\mathfrak{z}}} \leq \int_{\mathfrak{r}_\epsilon}^{\bar{\mathfrak{r}}} \int_{\mathfrak{z}_r}^{\bar{\mathfrak{z}}} \leq \int_{\mathfrak{r}_\epsilon}^{\bar{\mathfrak{r}}} \int_{\mathfrak{z}_r}^{\bar{\mathfrak{z}}} + \int_{\mathfrak{r}_\epsilon}^{\bar{\mathfrak{r}}} \epsilon'.$$

Then by theorem 27,

$$-\epsilon A + \int_{\mathfrak{z}_{r,\epsilon}}^{\bar{\mathfrak{z}}} \leq \int_{\mathfrak{r}_\epsilon}^{\bar{\mathfrak{r}}} \int_{\mathfrak{z}}^{\bar{\mathfrak{z}}} \leq \int_{\mathfrak{r}_\epsilon}^{\bar{\mathfrak{r}}} \int_{\mathfrak{z}}^{\bar{\mathfrak{z}}} \leq \int_{\mathfrak{z}_{r,\epsilon}}^{\bar{\mathfrak{z}}} + \epsilon A,$$

where A is some positive constant.

Letting $r \doteq \infty$, we get

$$-\epsilon A + \int_{\mathfrak{z}_\epsilon}^{\bar{\mathfrak{z}}} \leq \int_{\mathfrak{r}_\epsilon}^{\bar{\mathfrak{r}}} \int_{\mathfrak{z}}^{\bar{\mathfrak{z}}} \leq \int_{\mathfrak{r}_\epsilon}^{\bar{\mathfrak{r}}} \int_{\mathfrak{z}}^{\bar{\mathfrak{z}}} \leq \int_{\mathfrak{z}_\epsilon}^{\bar{\mathfrak{z}}} + \epsilon A.$$

Letting d , the norm of the division D of the x axis $\doteq 0$, we get (7). The case that $\mathfrak{r}$ is infinite is obtained now by a passage to the limit, as in theorem 29.

The theory of inversion of iterated integrals having unlimited fields of integration $\mathfrak{A}$ is contained in the theorems 29, 30, and analogous theorems which as remarked in § 5 may be readily deduced by reasoning similar to that employed to establish the theorems just cited. Lack of space requires their suppression here.

§ 8. Transformation of the variables.

Let D be a discrete partial aggregate of the region R . Let

$$T: \quad \mathfrak{r}_1 = \phi_1(x_1, \dots, x_m), \dots, \mathfrak{r}_m = \phi_m(x_1, \dots, x_m) \quad (J = \text{Det } T),^*$$

be regular in any inner partial region of R , exterior to D . Let $\mathfrak{D}$ the image of D , also be discrete. We shall say that the transformation T is *semi-regular in R* . At the points of D , J may vanish and the correspondence between D and $\mathfrak{D}$ may cease to be uniform. Moreover the ϕ 's and J may have points of finite or infinite discontinuity in D . The points D are called the *critical points of T* .

THEOREM 31.† Let

$$T: \quad \mathfrak{r}_1 = \phi_1(x_1, \dots, x_m), \dots, \mathfrak{r}_m = \phi_m(x_1, \dots, x_m) \quad (J = \text{Det } T),$$

define a semi-regular transformation in the region R , whose critical points

* The determinant of T may be denoted by $\text{Det } T$

† A similar theorem holds for the lower integrals.

we call D . Let X be an inner aggregate of R and let $\mathfrak{X}$ be its image. Let the singular points of f lie in $\mathfrak{D}$, the image of D . Then

$$(1) \quad \int_{\mathfrak{X}} f dx_1 \cdots dx_m = \int_X f |J| dx_1 \cdots dx_m,$$

provided either integral exists.

To fix the ideas, let the integral on the right exist. The reasoning we employ* applies to the other case. For simplicity let us introduce the following notation: A symbol as $\mathfrak{X}_{r,d}^*$ will denote a partial unmixed aggregate of $\mathfrak{X}_r^*$ exterior to the critical points such that

$$\overline{\text{cont } \mathfrak{X}_r^*} - \overline{\text{cont } \mathfrak{X}_{r,d}^*} \leq d.$$

We have to show, therefore, that having chosen ϵ , r_0 there exists a d_0 such that

$$(2) \quad \int_{\mathfrak{X}_{r,d}} = \int_X + \epsilon' \quad (|\epsilon'| < \epsilon),$$

for any $r \geq r_0$ and $d \leq d_0$.

Now by hypothesis there exists a pair s_0, e_0 , such that

$$(3) \quad \left| \int_X - \int_{\mathfrak{X}_{s,e}} \right| < \frac{\epsilon}{2},$$

for any $s \geq s_0, e \leq e_0$. Let U be one of the fields $X_{s,e}$, taken at pleasure, but then fixed. Let $\mathfrak{U}$ be its image. Let no point of $\mathfrak{U}$ be at a distance $> r_0$ from the origin. Let $\mathfrak{Z}$ denote the union of $\mathfrak{X}_{r,d}$ and $\mathfrak{U}$, $r \geq r_0$. Let $\mathfrak{Y} = \mathfrak{Z} - \mathfrak{X}_{r,d}$. Then

$$(4) \quad \int_{\mathfrak{Z}} = \int_{\mathfrak{X}_{r,d}} + \int_{\mathfrak{Y}}.$$

The image Z of $\mathfrak{Z}$ contains U and this is a field of the type $X_{s,e}$. Hence by (3)

$$(5) \quad \int_Z = \int_X + \epsilon'' \quad (|\epsilon''| < \frac{\epsilon}{2}).$$

Moreover, by *Lectures*, p. 554,

$$(6) \quad \int_{\mathfrak{Z}} = \int_Z.$$

Finally, $\mathfrak{U}$ being a fixed field, f is limited in $\mathfrak{U}$, and hence in $\mathfrak{Y}$. But $\text{cont } \mathfrak{Y} \doteq 0$, as $d \doteq 0$. Hence for some d_0 ,

$$(7) \quad \left| \int_{\mathfrak{Y}} \right| < \frac{\epsilon}{2},$$

* Cf. JORDAN, *Cours*, vol. 2, pp. 84-86.

for any $d \leq d_0$. Thus (4), (5), (6), (7), give (2), which was to be shown.

There are many variations of the preceding theorem; we mention only the following:

THEOREM 32.* *Let Δ be a discrete partial aggregate of the measurable field $\mathfrak{X}$. Let*

$$T: \quad x_1 = \phi_1(t_1, \dots, t_m), \dots, x_m = \phi_m(t_1, \dots, t_m) \quad (\text{Det } T = J),$$

be regular in any inner unmixed region in $\mathfrak{X}$, exterior to Δ . Let the image $\mathfrak{X}$ of $\mathfrak{X}$ be measurable and the image E of Δ be discrete. Let the singular points of $f(x_1, \dots, x_m)$ lie in E . Then

$$\int_{\mathfrak{X}} f dx_1 \dots dx_m = \int_{\mathfrak{X}} f |J| dt_1 \dots dt_m,$$

provided either integral exists.

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* A similar theorem holds for the lower integral.